

EXPLORATIONS OVER THE VIBRATING SURFACES OF TELEPHONIC DIAPHRAGMS UNDER SIMPLE IMPRESSED TONES.

BY A. E. KENNELLY AND H. O. TAYLOR.

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The following research was carried on, at the Massachusetts Institute of Technology, under an appropriation from the American Telephone & Telegraph Co. during the year 1914-1915. The experimental work was carried out at Pierce Hall, Harvard University.

The object of the investigation was to explore the amplitude of the small harmonic vibrations of a circular diaphragm of telephonic type, clamped around the edge, and to compare the observed values with those which had been already deduced mathematically. Hitherto, so far as we are aware, the amplitude of vibration of a telephone diaphragm has been determined only at one point on the surface, usually the center,¹ The observations here reported differ from those heretofore obtained, in extending over the entire surface of the diaphragms.

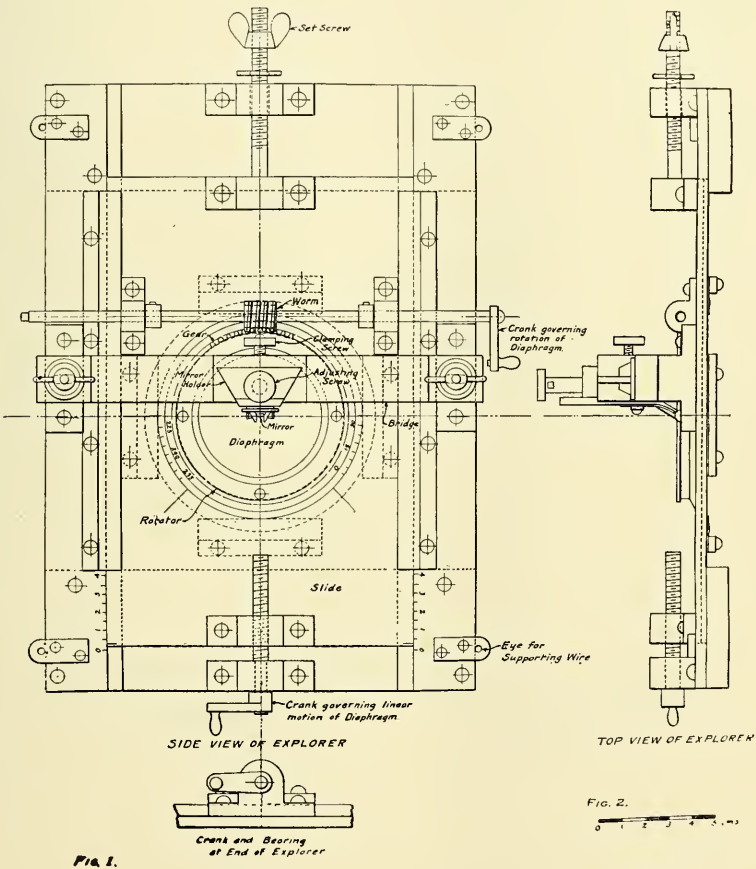
EXPLORING APPARATUS.

The exploring device, or "explorer," devised and constructed for this research, consists of a tiny triangular mirror fastened to a little phosphor-bronze stirrup strip, and having its point applied, by means of torsion in the strip, to the surface of the vibrating diaphragm at the point to be explored. The natural frequency of the mirror being much greater than that impressed on the diaphragm, the mirror is able to follow the vibrations of the latter, without breaking out of contact. The pressure exerted by the mirror on the diaphragm is so small as not materially to affect the diaphragm's vibration. A beam of light, reflected from the mirror on to a translucent scale, was thus set into vibrations synchronous with, and

¹ See Appended Bibliography, Nos. 2, 4, 7, 9 and 10.

proportional to, the vibrations of the diaphragm at the point of contact.

The vibration explorer is shown in side elevation at Fig. 1, in top view at Fig. 2, and in section, through center of the diaphragm, in Fig. 3. A fairly massive rectangular brass frame holds a plate sliding in grooves. The crank at the bottom of Fig. 1 controls this



sliding motion, with the aid of the set screw at the other end. At the center of the sliding plate is a circular frame, into which is clamped the diaphragm to be tested. The circular frame can be rotated in its own plane by means of the crank at the right hand of Fig. 1.

A stout brass bridge is fastened to the sides of the rectangular frame. At the center of this bridge is the mirror-holder shown in detail at Fig. 3. The mirror-holder slides in a groove provided in the bridge, and is clamped therein by a clamping screw *S*. A fine-motion screw *M* is also provided, for adjusting the position of the mirror. One turn of *M* advances the mirror 0.8 mm. ($1/32$ inch). By means of an auxiliary mirror fastened beneath the top of the screw *M*, the angle through which the screw is advanced may be

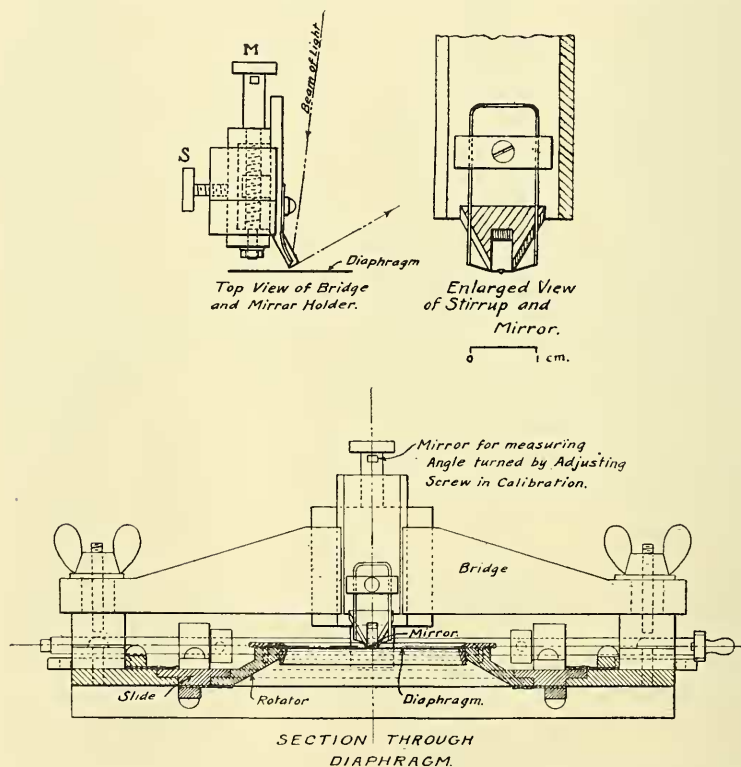


FIG. 3

APPARATUS FOR EXPLORING THE VIBRATION OF DIAPHRAGMS.

0 1 2 3 4 5 cms.

measured, for calibrating the indications of the instrument. Adjustment can be made to 1 deg. of rotation, or 2.2μ ; i. e. $\left(\frac{0.8}{360} \text{ mm.}\right)$ assuming that backlash is guarded against.

The construction of the apparatus is such, that the mirror is held at all times at the center of the brass rectangular frame; while by means of the two crank adjustments, the diaphragm to be explored can be moved so as to bring any part of its surface beneath the mirror. With the aid of the scales of distance and angle shown in Fig. 1, the position of the mirror with respect to the diaphragm can be adjusted and read off to polar coördinates (r , θ). The motion in r is controlled by the crank at the bottom, to 0.1 mm.; while the angular motion in θ is controlled by the crank at the side, to 1° , or less if desired. The slide is held in position by flat springs, attached to the rectangular frame, so as to keep the motion of the slide confined to its own plane. A similar construction is used with the circular frame. It is important that the plane of the diaphragm shall not be disturbed when either crank is operated. The weight of the whole explorer is 4.63 kgs. (10.2 lbs).

A magnified view of the mirror, and its stirrup frame, is shown at the top of Fig. 3. The mirror, of silvered glass, about 0.1 mm. thick, is cut in the shape of an equilateral triangle, about 1 mm. in length of side. One vertex of the mirror is applied to the surface of the diaphragm, and the mirror is fastened with sealing wax across a thin phosphor-bronze strip. This strip is approximately 3 mm. long between abutments, 0.02 mm. wide, and 0.013 mm. thick. The weight of the mirror is about 1 milligram, without varnish or sealing wax. Its natural frequency of vibration, as obtained photographically, is about 2,500 $\sim$. These little mirrors are apt to break off the stirrup strip; so that they have to be renewed and recalibrated occasionally. The pressure exerted on the diaphragm by the point of the mirror, as measured by an auxiliary test, is approximately 200 dynes (204 mgm. wt.). A pressure of this order seems to be desirable, so as to obtain a natural frequency of 2,500 $\sim$. If, however, explorations are confined to lower diaphragm frequencies, the natural frequency of the explorer mirror, and its pressure on the diaphragm, may be reduced accordingly.

The diaphragm to be explored is 5.4 cm. in diameter, and is placed in the circular frame. It is clamped tightly into this frame, with the ring clamp shown in Fig. 3, which had a radius of 2.62 cm. when no auxiliary clamping rings were used. The vibration explorer

is then suspended on wires from the ceiling, or other convenient support, in order to suppress building vibrations of high frequency; so as to support the explored diaphragm in a vertical plane. The mirror-holder is then advanced towards the diaphragm, and clamped by screw *S*. The mirror is now carefully brought into contact with the surface of the diaphragm by adjusting screw *M*. A picture of the explorer is presented in Fig. 4. The suspension wires *w**w*,

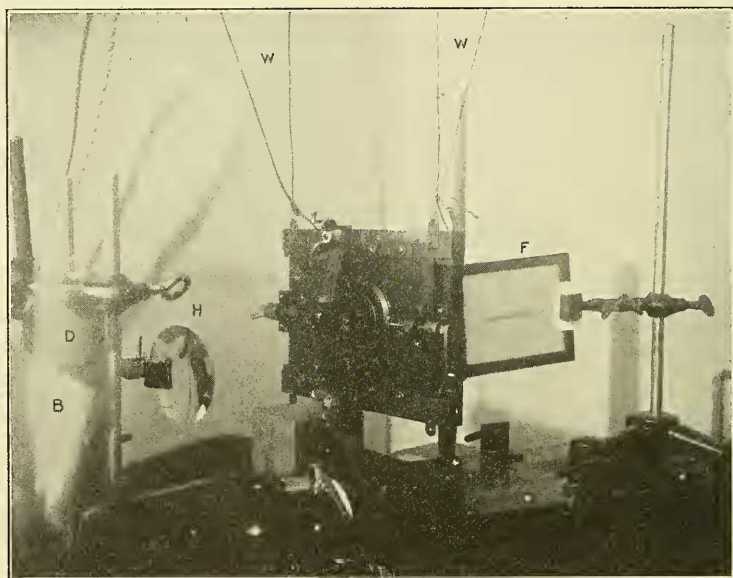


FIG. 4. Vibration Explorer in Booth.

support the instrument. The condensing and focusing lens *H* throws a narrow arc-light beam upon the exploring mirror, which reflects it on to the translucent graduated screen *F*. With the diaphragm at rest, the spot on this screen is a narrow, sharp, vertical, luminous strip. When the diaphragm is set in vibration, the mirror in contact with it vibrates synchronously, and the spot is spread into a luminous band, the limits of which are easily read on the graduated translucent scale. If the motions of diaphragm and mirror are simple harmonic motions, the luminous band shows no discontinuities of intensity. If, however, there is a complex harmonic motion in the diaphragm, the luminous band will show bright and dark

patches, either quiescent, or with beats. By means of the optical magnification of amplitude that can be effected with such an exploring mirror and scale, diaphragm vibrations of amplitude 0.1μ (*i. e.*, 10^{-5} cm.), or less, can be observed; although the precision of measurement falls off considerably, for a diaphragm amplitude below 0.5μ (half a micron).

OPTICAL SYSTEM.

The optical system employed with the vibration explorer is diagrammatically indicated in Fig. 5. The stereopticon arc-lamp *A* throws a powerful condensed beam of light on the pinhole *B*, in a brass vertical screen. A set of small powerful collimating lenses *C*

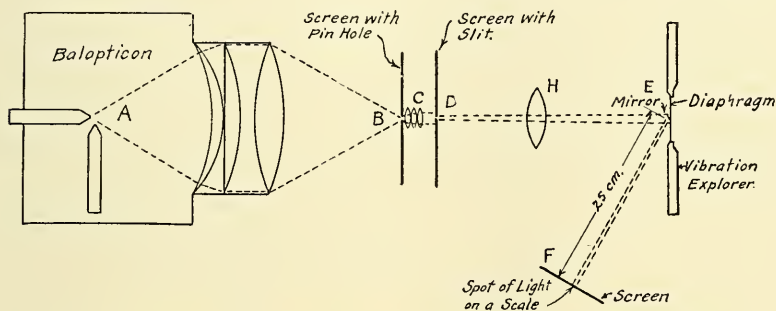


FIG. 5. Diagram of Optical System used with Vibration Explorer.

throws the nearly paralleled beam through the screen and slit *D*, as well as the focussing lens *H*, on the exploring mirror *E*, whence it is reflected to the translucent screen *F*, at a convenient distance, in this case 25 cm. An image of the slit in screen *D* is then sharply focused at *F*. In Fig. 6, it is indicated geometrically that the amplitude *e* of the diaphragm's displacement is equal to the continued product of the observed amplitude *d* of the luminous band, the ratio of *l* (the radius arm of the mirror), to $2L$ the double distance of the mirror from the screen, and the cosine of the angle ϕ between the radius arm of the mirror and the plane of the diaphragm. In order to avoid frequent changes in ϕ , it is desirable to keep constant the zero of the spot at the center of the graduated scale *F*, and with it the contacting angle of the mirror. The numerical

expression $M = 2L/l \cos \phi$ may be called the *magnification-factor* of the explorer. As ordinarily employed, $2L = 50$ cm., $l = 0.05$ cm., $\phi = 45^\circ$ approximately, or $\cos \phi = 0.7$; so that $M = 1,400$ approximately, varying in different sets of measurements between

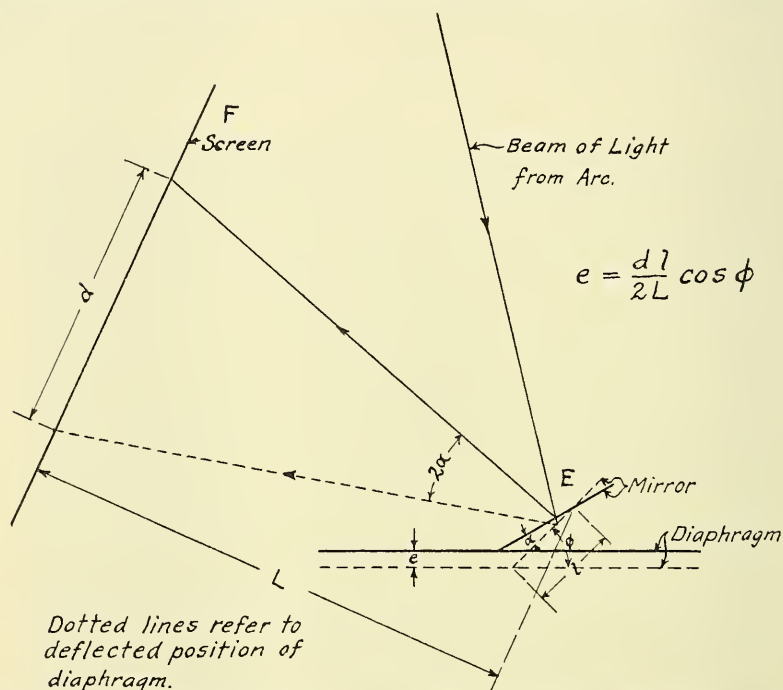


FIG. 6. Diagram Showing Action between Mirror and Diaphragm in Explorer.

800 and 1,500. Had it been necessary, this magnification-factor might have been considerably increased, by increasing the distance L between mirror and scale; although the reduction in luminous-spot intensity, at increasing ranges, prevents the precision of the observations from increasing in the same proportion as the magnification-factor. At $L = 25$ cm., the amplitude of luminous band could be read to 0.1 mm. on the graduated translucent scale F .

SOURCE OF DIAPHRAGM VIBRATIONS.

Two sources of vibrations were used in different series of tests. (1) acoustic, (2) electromagnetic.

(1) The acoustic vibrations were supplied from one of a series of small organ-pipes, giving fairly simple musical tones between C_2 of 128 $\sim$, and C_6 of 2,048 $\sim$. The organ-pipe selected was mounted vertically in a block on the table, at the back of the vibration explorer, and supplied with air at constant pressure (about 18 cm. of water) from a pneumatic tank. The whole apparatus was placed inside a sound-damping wooden-frame booth (274 cm. $\times$ 183 cm. $\times$ 214 cm. high), lined on the inside with hair-felt, 2.5 cm. thick, surfaced with thin cloth. The observer, after turning on the air to the organ-pipe, observed the amplitude of the luminous band on the translucent screen F , Figs. 4 to 6, as the mirror was applied to different successive points on the diaphragm.

(2) The clamping ring of the diaphragm in the explorer was chosen of such dimensions that a standard telephone receiver could be substituted for it. In this case, a steel diaphragm had to be employed. The telephone was then operated by a feeble measured alternating current (2.0 milliamperes) obtained from a Vreeland mercury-arc oscillator having a frequency adjustable, by successive steps, between 430 $\sim$ and 2,500 $\sim$.²

EXPLORATION WITH DIAPHRAGM No. I.

Diaphragm No. I was a telephone-receiver diaphragm of steel, jappanned on one side. Its dimensions are given in Table III. The diaphragm was clamped, around the boundary, between opposing circular knife-edges.

TABLE I.

VIBRATION AMPLITUDES OVER DIAPHRAGM No. I, AT FREQUENCY 608 $\sim$, FOR NINE DIFFERENT AZIMUTHS θ , AND SEVEN DIFFERENT RADIAL DISTANCES r .

Radial Distance, r Cm.	Vibration Amplitude Observed with Explorer (Microns) at Different Azimuths θ .								
	0° μ	40° μ	80° μ	120° μ	160° μ	200° μ	240° μ	280° μ	320° μ
-0.08	13.8	12.1	11.7	14.0	13.4	10.4	10.8	13.3	12.0
+0.31	12.7	10.9	11.3	12.3	12.7	9.7	10.6	12.6	11.6
+0.69	9.7	8.0	8.9	10.4	10.4	8.1	8.8	11.5	9.5
+1.06	6.6	5.9	6.4	6.8	7.0	6.2	6.4	7.1	6.8
+1.44	4.2	3.6	4.2	4.7	4.5	4.1	4.2	4.9	4.3
+1.82	2.5	2.2	2.1	2.5	2.5	2.3	2.4	2.5	2.4
+2.20	0.9	0.8	0.9	0.9	0.9	0.7	0.7	0.9	0.8
+2.54	0	0	0	0	0	0	0	0	0

² See Bibliography No. 5.

An organ-pipe of D_4^* (608 $\sim$) was set up with its lip 5 cm. from the back of the diaphragm. An exploration was then made over the surface, at points differing by 40° in azimuth θ , and at successive increases in radius of about 3.3 mm. (7 steps in r , and 9 steps in θ , or 63 observations in all.) The preceding table gives the observed amplitudes of vibration deduced from the scale-deflections, with a magnification factor of $M=1,180$.

It will be seen from the above table, that at any particular radius r , measured from the center of the diaphragm, the amplitudes at varying azimuths θ are substantially equal. The irregularities are small, but nevertheless seem larger than can be accounted for by errors in observations and are, perhaps, due to irregularities in the diaphragm. Fig. 7 shows the contour lines of vibration-amplitude in microns, the maximum amplitude being at or near the center, and amounting to 14μ . Such vibration amplitudes are larger than were usually obtained, and were specially reinforced in this case, in order to secure large deflections. It will be seen from the contour diagram, that the diaphragm was vibrating with its fundamental or gravest mode of motion; *i. e.*, a motion to-and-fro as a whole, without either nodal diameters or nodal circles. It is known that a circular diaphragm, clamped at the edge, is capable of vibrating in an indefinitely large number of ways, according to the

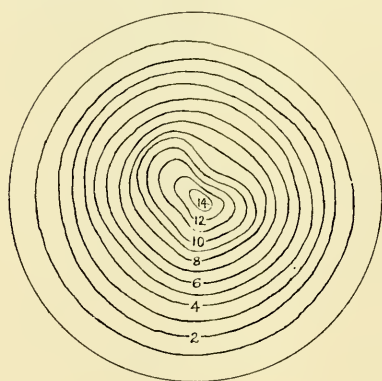


FIG. 7.

VIBRATION AMPLITUDE
IN MICRONS.
608 $\sim$
DIAPHRAGM No. 1.

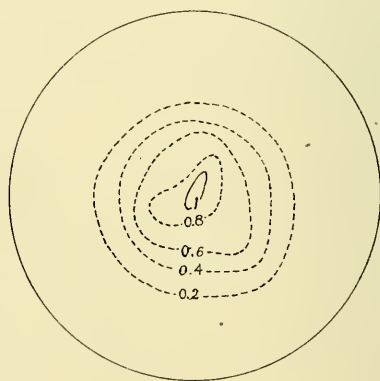


FIG. 8.

VIBRATION AMPLITUDE
IN MICRONS.
2100 $\sim$
DIAPHRAGM No. 1.

number of nodal circles, and also according to the number of nodal diameters present.³

A similar exploration was made over the diaphragm, with acoustic excitation from an organ-pipe giving C_6 (2,100 $\sim$). Here the points of observation were in steps of about 3.3 mm. in r , and in steps of 40° in θ , as before, with magnification-factor, $M=1,265$.

TABLE II.

VIBRATION AMPLITUDES OVER DIAPHRAGM NO. 1, AT FREQUENCY 2,100 $\sim$, FOR NINE DIFFERENT AZIMUTHS θ , AND SEVEN DIFFERENT RADIAL DISTANCES r , FIVE ONLY GIVING READABLE DEFLECTIONS.

Radial Distance, r cm.	Vibration Amplitude Observed, with Explorer, at Different Azimuths θ .								
	0° μ	40° μ	80° μ	120° μ	160° μ	200° μ	240° μ	280° μ	320° μ
-0.08	0.70	0.95	0.95	0.95	0.87	1.02	0.95	0.87	0.95
+0.31	0.70	0.87	0.87	0.78	0.78	1.02	0.87	0.78	0.87
+0.69	0.62	0.78	0.78	0.62	0.62	0.78	0.62	0.62	0.78
+1.06	0.31	0.39	0.39	0.39	0.39	0.39	0.39	0.39	0.62
+1.44	0.16	0.16	0.16	0.08	0.08	0.08	0.08	0.08	0.23
+1.82	—	—	—	—	—	—	—	—	—
+2.20	—	—	—	—	—	—	—	—	—
+2.54	0	0	0	0	0	0	0	0	0

The vibration contour-lines for this case are given in Fig. 8. Here again it is seen that, setting aside irregularities in the diaphragm, and allowing for errors of observation (which are more noticeable with the small amplitudes of higher pitch), the mode of vibration is essentially fundamental, since there are no perceptible nodal circles or nodal diameters.

Having thus ascertained that both at pitch $D_4^\sharp$ (608 $\sim$), and at $\bar{C}_6$ (2,048 $\sim$), the first mode of vibration was presented, a series of careful explorations were made at a number of intermediate pitches. These likewise all showed the first or fundamental mode of vibration. See Table IIA.

Observations were also made, at organ-pipe frequencies down to 128 $\sim$. Explorations would be very difficult to obtain on this diaphragm at such low frequencies, owing to the small vibration amplitudes produced; but the indications were that the fundamental mode of vibration was maintained throughout.

³ See Appendix I.

The conclusion, therefore, seems warranted that, for this particular steel telephone diaphragm, acoustically excited to frequencies as high as $2,100 \sim$, the fundamental mode of vibration is the only one that is maintained. If any higher modes of motion were present, they were too faint to be discerned. This does not mean that higher modes of motion could not be produced by any kind of excitation within the above ranges of frequency. The effects of very powerful vibrations were not investigated.

Since the natural frequency of this diaphragm, with flat clamping, was observed to be $n_0 = 824 \sim$, and since, according to Bessel-Function theory, the natural frequency of the second mode of motion should be $2.09n_0$, we should naturally expect to find this second mode of motion appearing at and above $1,720 \sim$. Its non-appearance may have been due to the uniformity of acoustic impressed force over the surface, which would tend to favor the first rather than the second mode of forced vibration.

The vibration-amplitude of the diaphragm was found to vary widely with the pitch of the exciting source. At or near the natural

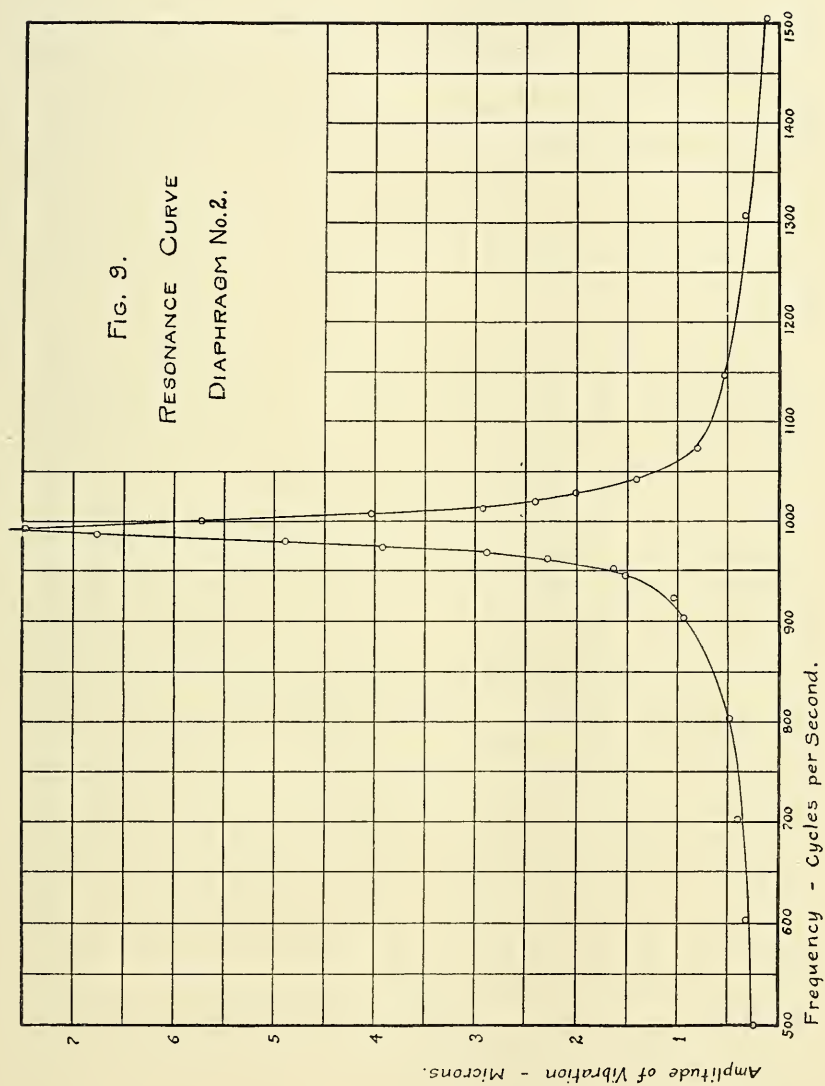
TABLE II*A*.

SHOWING FUNDAMENTAL MODE OF VIBRATION MAINTAINED FOR A RANGE OF FREQUENCIES FROM $400 \sim$ TO $1,800 \sim$.

Amplitudes of Vibration in Microns (μ) along Radius of Diaphragm No. 1, Flat-Clamped. $n_0 = 704 \sim$.

Radial Distance, r Cm.	Frequency of Vibration.						
	$400 \sim$ μ	$500 \sim$ μ	$750 \sim$ μ	$1,000 \sim$ μ	$1,250 \sim$ μ	$1,500 \sim$ μ	$1,800 \sim$ μ
.04	.8	1.6	7.8	1.3	1.3	.9	.3
.29	.8	1.6	7.7	1.3	1.2	.9	.3
.54	.7	1.5	7.2	1.1	1.1	.8	.3
.79	.6	1.4	6.3	.9	.9	.6	.2
1.04	.5	1.3	5.3	.8	.8	.5	.2
1.29	.4	1.0	3.8	.5	.6	.3	.1
1.55	.2	.7	3.0	.3	.4	.2	+
1.79	.1	.4	1.9	.2	.2	.1	+
2.04	+	.3	1.2	.1	.1	+	—
2.30	—	.1	.8	—	+	—	—
2.65	0	0	0	0	0	0	0

fundamental frequency of the diaphragm, the amplitude of the vibratory response was a maximum. Either above or below this



resonant frequency, the amplitude of vibration, shown by the explorer, fell off very markedly. The curve of relative amplitude at different frequencies is indicated in Fig. 9. It will be seen that when exciting the diaphragm with vibrations remote from the resonant frequency in either direction, the amplitude becomes so small that the degree of precision which may be obtainable near resonance is impossible to secure. The outline theory for this resonance curve, Fig. 9, is given in Appendix II. It is shown that if we multiply the successive ordinates by $\omega = 2\pi n$, the resulting velocity-values correspond to vector chords on a certain velocity circle.

Fig. 1B of Appendix I. gives the graph of the explored vibration amplitudes, at successive radial distances from the center of diaphragm No. 1, for the frequency 896 $\sim$. It will be seen that the amplitude falls off smoothly from a maximum at or near the center ($r=0$), to zero at the flat-clamped edge ($r=2.62$). The application of Rayleigh's theory of free vibration to these curves is given in Appendix I. In general, the agreement between the acoustically forced amplitudes and theoretically computed free amplitudes was satisfactory.

At or near the resonant frequency, or natural frequency of a diaphragm, especially when its damping coefficient is small, so that the resonance is sharp, a small change either in impressed frequency, or in the constants of the diaphragm due to change of temperature, may have an appreciable influence upon the amplitude of vibration. In other words, although the observed amplitudes are relatively large, and the precision of measurement is seemingly high, yet the system is in a virtually unstable condition. Consequently, although there is no reason to suppose that the conditions at resonance differ from those off resonance, nevertheless, when a reliable and reproducible set of observations of amplitude distribution is desired, it is advisable to select a frequency not too close to resonance, or say of about half the resonant amplitude.

APPLICATION OF CIRCULAR VELOCITY-DIAGRAM THEORY TO
RESULTS OF EXPLORATIONS.

It is shown in the first-approximation theory of Appendix II., that the behavior at the center of a flat-clamped circular diaphragm, subject to constant vibro-motive force of varying frequency, can be completely predicated, if three constants of the diaphragm are known;⁴ namely,

- (1) the "equivalent mass" m (gm.),
- (2) the elastic constant s (dynes per cm. of displacement at center),
- (3) the mechanical resistance r (dynes per unit velocity at center).

All these three constants can be obtained, for an acoustically excited diaphragm, with the aid of the vibration explorer.

DETERMINATION OF m .

In order to determine the equivalent mass of a diaphragm, it is necessary to know the distribution of amplitude over the entire vibrating surface. As is shown in Appendix III., when the distribution of amplitude conforms regularly with the Rayleigh formula, it would appear that the equivalent mass is 0.183 times the mass of the circular vibrating plate. If, however, the distribution of amplitude is irregular, such as may be produced by bipolar electromagnetic excitation of a telephone-receiver diaphragm, the coefficient 0.183 cannot be depended upon, and the proper coefficient must be determined by some process of quadrature, such as Appendix III. describes.

THE ELASTIC CONSTANT s .

The constant s is the inferred elastic resisting force, which, acting perpendicularly upon the diaphragm's equivalent mass (at its center), would produce the same effect upon the vibratory motion as the distributed elastic forces produce upon the diaphragm's distributed mass, in the presence of the particular impressed force distribution. The simplest way to find s is to measure the natural fundamental frequency n_0 of the diaphragm, by exciting it with an

⁴ See Bibliography No. 8.

organ-pipe of adjustable pitch, tuned to produce the maximum vibratory amplitude at the center. As shown in Appendix II., the constant s is then the product of the equivalent mass m and the square of the resonant angular velocity ω_0 .

A series of statical measurements were made, by applying small tensions f_s , by means of a calibrated spring, to the center of the diaphragm, and observing, with the aid of the explorer, the central displacements w_s , thereby produced. It was found, as might be expected, that the ratio of f_s to w_s was constant, so long as the latter did not exceed 18μ . Moreover, the value of s obtained from f_s/w_s was approximately the same as that obtained from formula (9), App. II. This static method of finding s , however, is inferior to the resonance method, because precise static measurements are difficult to obtain. The application of electro-magnetic excitation to a steel diaphragm also imposes residual stresses, which make the use of the static method unreliable.

THE MECHANICAL RESISTANCE r .

The constant r was measured, with the explorer, by photographing the decay curve of vibration amplitude on a moving photographic film, when the diaphragm was tapped at the center, and allowed to return to the equilibrium position under its own damping forces. It is shown in Appendix II., that the resistance r is twice the natural frequency multiplied by the equivalent mass and the logarithmic decrement. Fig. 10 is a tracing from a photograph of



FIG. 10. Tracing from Photograph of Decay Curve. Diaphragm No. 1.

the curve of decay. A small camera, represented in Fig. 11, was set up in front of the explorer, containing a photographic film wrapped around a metal drum. The drum was motor driven at a peripheral speed of approximately 4 meters per second, and the shutter was opened at the time of tapping the diaphragm. The logarithmic decrement of this curve is 0.184, at the frequency of $824 \sim$; so that with an equivalent mass m of 1.09 gm. the value of

r becomes 328 dynes per cm. per sec. The precision in measuring r by this method is relatively low, owing to the difficulty in measuring the successive amplitudes with accuracy, on a curve of such small dimensions.

Since, as is shown in Appendix II., a circular diaphragm, in its fundamental mode of motion, ordinarily develops a circular graph of velocity, at varying impressed frequency, with constant vibro-

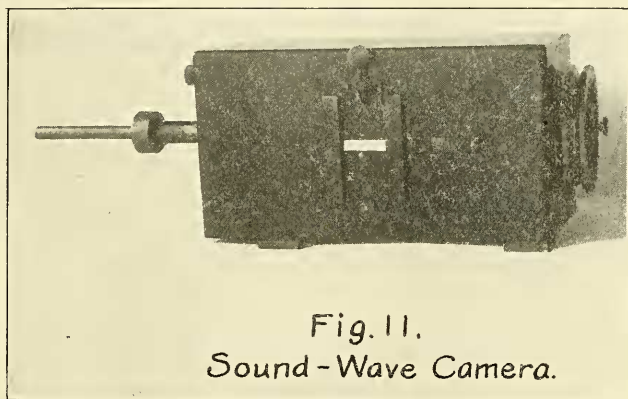


Fig. 11.
Sound-Wave Camera.

motive force, the plan has suggested itself, in the course of this research, to use the circle-velocity diagram of a diaphragm for comparing the vibro-motive forces (vmf.'s) of different organ-pipes. In this connection, the vmf. of a pipe at the exploring diaphragm, may be defined as its harmonically varying pressure $f = F\epsilon^{i\omega t}$ (dynes) produced, at the diaphragm, by the pipe, under the geometrical conditions of the system, including acoustic reflections from walls, or other objects in the room, on both surfaces of the exploring diaphragm. In the simplest, or standard, geometrical condition, the *standard* vmf., which is proportional to the square root of the sound intensity⁵ at the diaphragm, would be observed in free space, with the orifice of the pipe facing the diaphragm at a definite distance, and with the diaphragm perpendicular to the line joining them. It is our understanding that there is, as yet, no simple published method of measuring the vmf. of organ pipes, of different

⁵ Bibliography 6, Barton, "Text Book of Sound," p. 211, par. 146. Macmillan Co., 1908.

sizes or pitches, at a definite distance from their orifices. If, in a given geometrical environment, pipes of different pitches are set up, in succession, at the same position with respect to the exploring diaphragm, then the observed amplitudes, multiplied by the respective values of ω , should lie on the velocity circle-diagram, if the vmf.'s of the pipes are the same; assuming that the fundamental mode of vibration is produced, that the constants of the diaphragm remain unchanged, and that the overtones of the pipes are negligibly small. The vector departures from the circle diagram would then indicate the inequalities in vmf.'s.

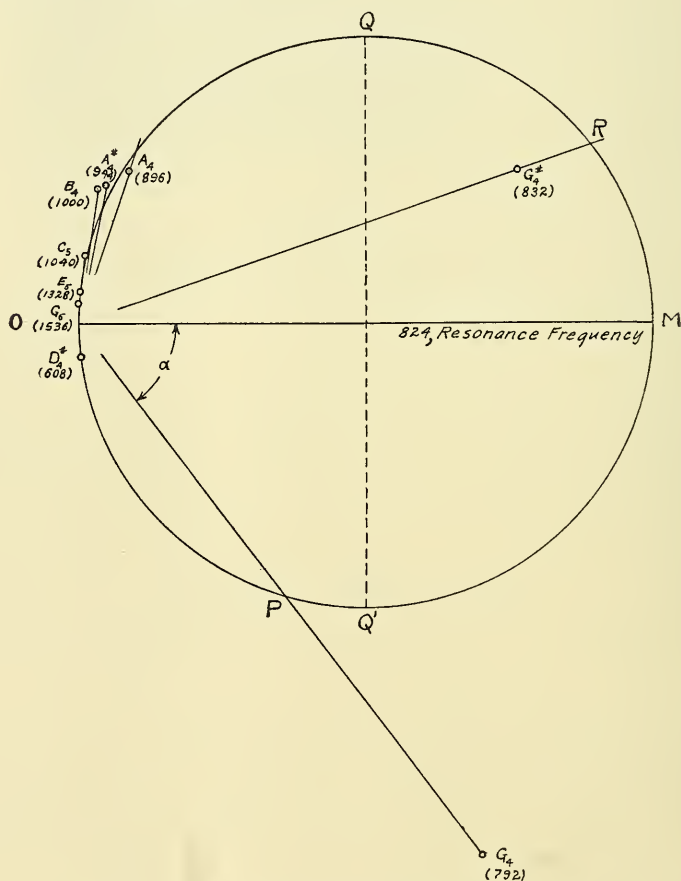


FIG. 12. Diagram Showing Strengths of Organ Pipes Given by the Vibration of a Diaphragm.

Fig. 12 is an inverted velocity-circle diagram for Diaphragm No. 1, based upon its measured values of m , r and s . If we take the diametral velocity OM as 5 cm. per sec., with $r=328$ dynes per cm./sec., then the vmf. which, in the particular environment of the experiment, produced this velocity, would be 1,640 dynes, maximum cyclic value. The particular pipe $G_4(792 \sim)$, gave an observed amplitude at the diaphragm center, which, multiplied by $\omega = 2\pi \times 792$, gives the line OG_4 along the chord OP . The phase-angle α must be obtained by considering the mechanical reactance as in (4), App. II. If the vmf. of this pipe were the same as that which produced OM , this point G_4 , would lie on the circle. Consequently, the vmf. of the pipe G_4 is to that of the pipe producing resonance, in the ratio OG_4/OP . Similarly, the vmf. of the pipe $G_4^*(832 \sim)$, is less than that producing the resonant velocity, in the ratio OG_4^*/OR . It is evident that the range of any one diaphragm, for the precise comparison of vmf.'s from organ-pipes of different pitch, is somewhat limited. In the case presented, it would not exceed one octave, since the chords far from the resonant diameter become so short. By selecting a diaphragm of relatively large damping constant $\Delta=r/2m$, this range can be increased. In fact, the range in ω between the quadrantal points QQ' on the velocity circle, is numerically equal to r/m , or twice the damping constant.

A succession of calibrated diaphragms with overlapping ranges might be employed to cover the musical scale. The writers have not attempted to compare organ-pipes for standard vmf. in this manner. The measurements might have to be made out-of-doors. In the sound-absorbing room in which this research was carried on, the effect of sound reflections from walls and other objects prevented any standard comparisons of vmf. from being made.

EXPLORATIONS WITH ELECTROMAGNETICALLY EXCITED DIAPHRAGMS.

In order to ascertain the effects of exciting a steel diaphragm (No. 2) electromagnetically, a No. 144 Western Electric Bell telephone receiver was screwed into the explorer, behind the diaphragm, so as to obtain the ordinary air-gap between the diaphragm and its two poles. The cap or screw-cover of the ordinary telephone re-

ceiver was here absent. Alternating current of 2 milliamperes (root-mean-square) was supplied from a Vreeland oscillator, giving a close approximation to a pure sine wave, and in connection with a Rayleigh bridge, for the simultaneous measurement of both the resistance and inductance of the telephone receiver, at 32 frequencies varying between 429 and 2,040 $\sim$. Explorations were

VIBRATION CONTOURS DIAPHRAGM No. 2.

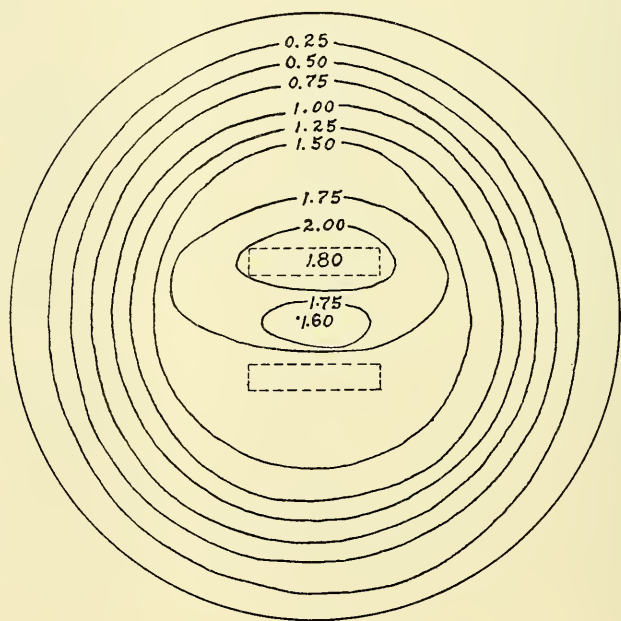
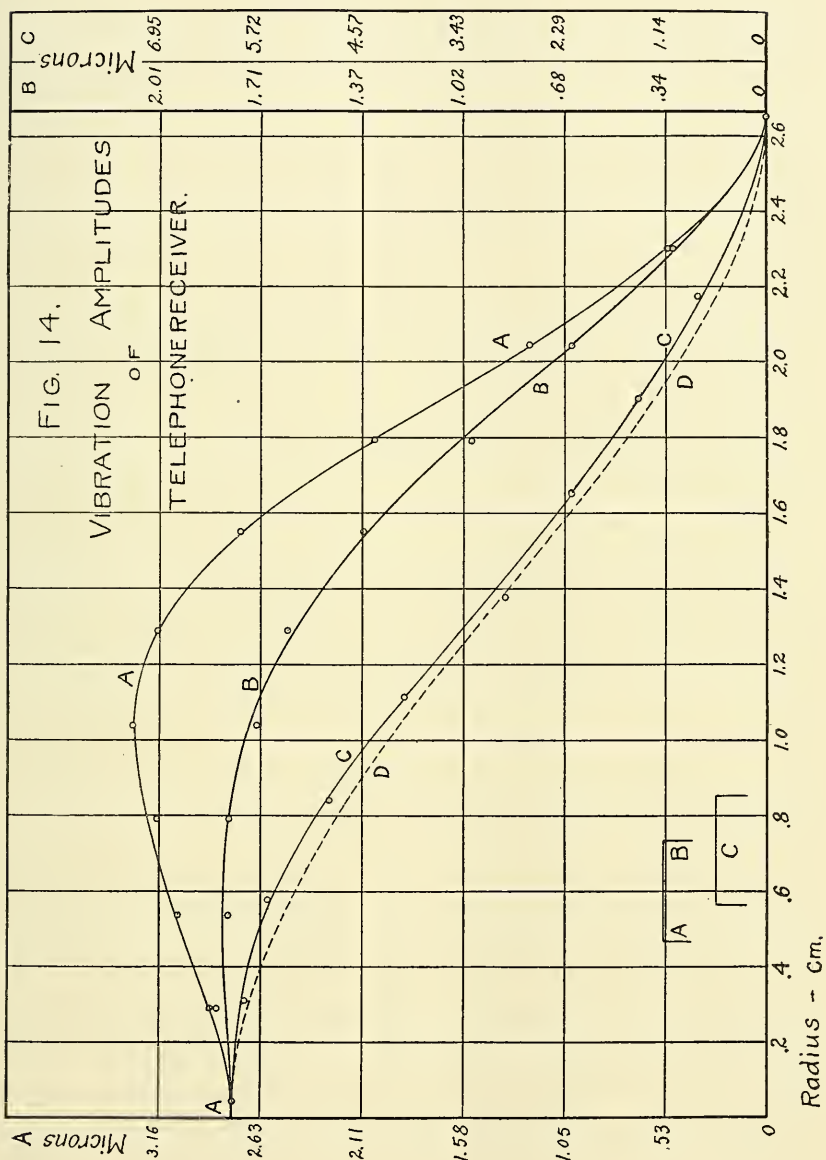


Fig. 13.

made at two frequencies; one, the resonant frequency of 992 $\sim$, and the other slightly below this, or 974 $\sim$. The contour lines for the latter case are presented in Fig. 13, where the outlines of the



two magnetic poles are indicated in dotted lines. It will be seen that while the mode of motion is essentially fundamental, the amplitude is not a maximum at the center, as in the ordinary acoustic case. The maximum amplitude of 2.0μ is reached in an elliptical loop embracing the pole at the top. Inside this loop, and immediately over the pole, the amplitude falls off to 1.8μ . Over the pole underneath, the amplitude is about 1.7μ , but there appears to be a slight diminution between the poles. If the geometrical and magnetic conditions of the bipolar system were perfectly symmetrical, these dissymmetries would presumably disappear.

The curves of mean amplitude against radial distance are presented in Fig. 14. The curve *AAA* corresponds to that found at resonance, and shows that the amplitude is far from being a maximum at the center of the diaphragm, owing to the attractive forces being established over polar areas on each side of the center. The coefficient of equivalent mass for this curve is over 0.5.

The curve *ABB* gives the corresponding distribution of mean azimuthal amplitude for the frequency of $974\sim$. The swelling of the amplitude over the poles is less marked in this case, and does not materially exceed that at the center. The equivalent mass coefficient for this curve is 0.36, or about double that for the Rayleigh-Bessel curve case, which is indicated by *ADD*. The curve *ACC* gives the distribution of mean amplitude in radial distance, for another steel diaphragm (No. 3) in a bipolar telephone receiver, at the resonant frequency of $1,020\sim$.

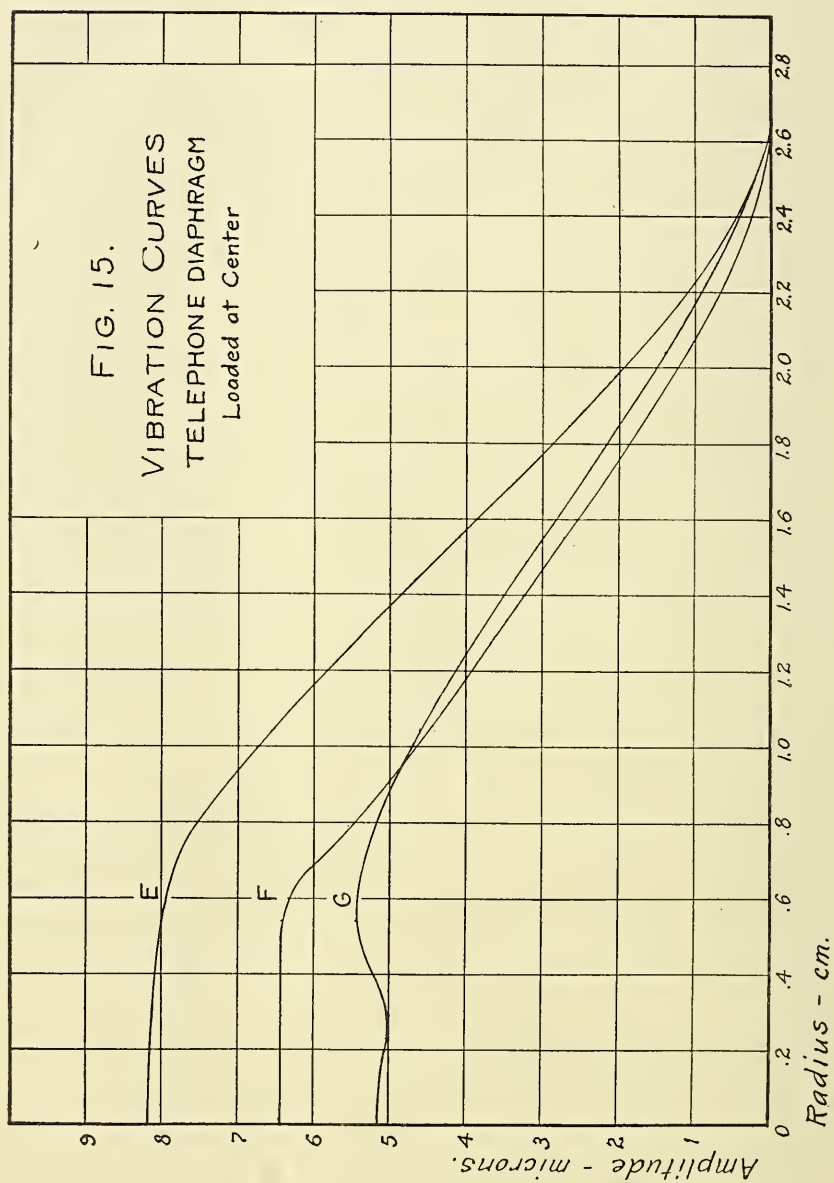
For both steel diaphragms Nos. 2 and 3, a series of central amplitude measurements were made, with the explorer, at constant alternating-current excitation, but adjustably varied frequency. Simultaneous measurements were made by Mr. H. A. Affel, of the resistance and inductance of the telephone-receiver coils, with the diaphragm both free and damped. The explorer measurements in both cases satisfactorily checked the electrically deduced velocity-circle diagrams. It is proposed to report upon the electrical measurements in another paper. Moreover, starting with the amplitudes, measured at the center of the diaphragm, in curves *A* and *C* of Fig. 14, the equivalent masses of the diaphragms, computed from the electrical measurements, agreed, within a few per cent., with those found by integrating curves *A* and *C*.

TEMPERATURE EFFECTS.

It was found that changes of temperature in the air surrounding a diaphragm had a marked effect, both upon its resonance frequency, and upon its amplitudes at any frequency. The curves representing w against r , were apt to differ appreciably in outline from day to day. The degree of tightness of clamping also had a marked effect in these measurements. In general, such disturbances due to temperature and clamping, are likely to introduce tensions in the substance of the diaphragm, and to cause some of the characteristics of vibrating membranes to be superposed upon those of a vibrating plate. It is, therefore, desirable that the clamping should be effected tightly, and that the measurements should then be made before the temperature has changed. Strictly speaking, the Rayleigh theory shows that there must be a marked difference in both the resonance frequency and in the distribution of amplitudes, if the diaphragm is clamped between circular knife edges, instead of between circular flat rings at the boundary. The experiments have shown that flat-ring clamping is more likely to give consistent results than knife-edge clamping. These clamping difficulties are accentuated in thin glass diaphragms, for the boundary supporting of which, a special technique had to be developed.

EXPLORATION OF THIN GLASS DIAPHRAGMS.

From a number of thin glass diaphragms, one Diaphragm No. 4, was selected, on account of its uniformity in thickness. See Table III. It was found very difficult to obtain uniform results with this in the explorer, owing to the above mentioned troubles with clamping. Finally, the glass diaphragm was cemented, with water glass, to a boundary ring of glass, and this was lightly supported between the clamping rings of the explorer. The diaphragm was then excited acoustically by organ-pipes. The natural pitch of the diaphragm was found to be $492 \sim$, in the fundamental mode. On raising the frequency, the mode of motion was found to change suddenly, at $968 \sim$, to that of a single nodal diameter, the two halves of the diaphragm then vibrating harmonically in opposite phases. This mode of motion continued until the frequency reached $1,696 \sim$,



when the nodal diameter disappeared and gave place to a single nodal circle. The ratios of the above three frequencies are $1:1.97:3.44$; whereas, according to the Bessel-function theory, they should be $1:2.09:3.91$. The discrepancies may readily be accounted for by imperfections in boundary support, or by temperature effects. Small changes in clamping were found to exercise a marked influence on these ratios.

LOADING OF DIAPHRAGM.

In the determination of m , r and s , by electrical impedance measurements,⁶ only two quantitative relations between these three constants naturally present themselves; whereas, for the evaluation of these three unknowns, three independent quantitative relations must be experimentally obtained. It had been hoped to derive the missing third equation, by applying a small known load-mass at the center of the diaphragm, and by repeating the electrical measurements with this load in place. Electrical experiments showed, however, that while, occasionally, consistent results were obtained in this way, more often the results were discordant. The reason for the discordance has been shown, from explorations of the diaphragm, to be due to a distortion of the amplitude curves; whereby the equivalent mass of the loaded diaphragm is no longer the same as when unloaded.

These conditions are exhibited in the curves of Fig. 15. E shows the w , r curve, for an unloaded telephonic steel diaphragm, excited acoustically at $n=904\sim$, its natural frequency being $n_0=832\sim$. The corresponding curve F is for the same diaphragm, after being loaded at the center by a small brass cylinder of 0.536 gm. at $n=816\sim$, its new natural frequency being $n_0=696\sim$. After increasing the load to 1.08 gm., the new curve is shown at G ($n=660\sim$, $n_0=616\sim$). The shapes of these three curves E , F and G , being so different, it is evident that the equivalent mass of the diaphragm by itself cannot be regarded as constant.

The authors are indebted to Dr. Geo. A. Campbell for a number of valuable suggestions which he made after having read the MSS. of this paper; also to Professor W. C. Sabine for very useful suggestions, during the course of the research.

⁶ Bibliography No. 8.

SUMMARY.

1. The distribution of amplitudes over small circular telephonic diaphragms, under simple impressed vibrations, has been measured, it is believed for the first time, by means of a new and specially constructed vibration-explorer.

2. The simple vibrations of the small steel circular diaphragms, used in telephonic receivers, appear to belong to the fundamental mode, within the ordinary telephonic range of intensity and frequency up to 2,000 $\sim$, with the distribution of impressed forces here described.

3. The explorations have confirmed the working theory of the velocity-circle diagram for such vibrations, and have afforded means of determining the three constants m , r and s , in that theory, for acoustically excited vibrations.

4. In the resonant condition, exploration is somewhat uncertain, owing to slight instability in the vibratory behavior of the diaphragm.

5. The distribution of forced amplitude at varying radial distances, has been found to compare well with the Rayleigh theory of freely vibrating plates, when good flat clamping around the edge can be secured, and with acoustic excitation. The coefficient of equivalent mass appears to be 0.183 for such a case. With electromagnetic excitation, the amplitude distribution may be very different and the coefficient is ordinarily increased.

6. Loading a diaphragm with a small mass at the center, decreases its natural frequency, and tends to reduce the amplitude of vibration at the center, with a relative increase at outlying points; so that the equivalent mass of the diaphragm, considered by itself, is apt to be changed.

7. A means is suggested, based on the velocity-circle diagram, for comparing the acoustic intensities of organ-pipes of different pitches.

8. The distribution of amplitudes over the surface of a steel receiving-telephone diaphragm, with bipolar electromagnetic excitation, was found to be of fundamental mode, but with a tendency to form two maxima, one over each pole.

9. In some small, thin, glass diaphragms, three modes of vibra-

tory motion were observed, in the range of acoustic impressed frequency up to 1,700 ~.

TABLE III.
FLAT CIRCULAR DIAPHRAGMS.

No.	Material.	Diameter, Cm.	Thickness* Over Japan, Cm.	Mass, Gm.	Natural Frequency ~.
1	Steel japanned.....	5.4	0.038	5.615	824
2	Steel japanned.....	5.52	0.0399	5.979	992
3	Steel japanned.....	5.48	0.031	4.181	1020
4	Glass.....	5.4	0.0108	0.6548	492

APPENDIX I.

Application of Bessel-Function Theory to a Diaphragm Vibrating in its Fundamental Mode.

Referring to Lord Rayleigh's "Theory of Sound," Vol. I, page 352, the formula for the instantaneous amplitude of free vibration in a flat plate is,

$$w_n = P\{J_n(kr) + \lambda J_n(ikr)\} \cos(n\theta + \alpha_n) \cdot \cos(\omega t + e) \quad \text{cm.}, (1)$$

where subscript n = the number of nodal diameters (numeric),

w_n = instantaneous amplitude at a point on the diaphragm whose polar coördinates are r cm., θ radians (cm.)

P = constant of amplitude-magnitude (cm.),

k = a constant of the material defined by:

$$k = \sqrt{\omega}/c \quad (\text{cm.}^{-1}),$$

c = a constant of the material defined by:

$$c = \sqrt[4]{\frac{qb^2}{12\rho(1 - \sigma^2)}} \quad (\text{cm./sec.}^{\frac{1}{2}}),$$

q = Young's modulus for the diaphragm material (dyne/cm.²),

ρ = density of the diaphragm material (gms./cm.³),

σ = Poisson's ratio for the diaphragm material (numeric),

b = thickness of the diaphragm (cm.),

λ = a constant satisfying boundary conditions (numeric),

J_n = a Bessel's Function of the n th order (numeric),

$$i = \sqrt{-1},$$

* Thickness of japan 0.0074 cm.

α_n = a phase-angle measured around the diaphragm (radians),
 $\omega = 2\pi n$ = angular velocity of vibrating motion (radians/sec.),
 n = frequency of diaphragm vibration (cycles/sec.),
 t = time elapsed from a given epoch (seconds),
 e = a time-phase determined by the epoch (seconds),
 a = radius of the diaphragm (cm.).

For the fundamental mode of motion, $n=0$; or there must be no nodal diameters. Consequently (1) reduces to:

$$w_0 = P \{ J_0(kr) + \lambda J_0(ikr) \} \cos(\omega t + e) \quad \text{cm.} \quad (2)$$

Here the amplitude of vibration at any point w_0 , ceases to be a function of θ , and depends only on Bessel functions of r . Since we shall consider only the fundamental mode of vibration in what follows, the subscript will be unnecessary, and we may substitute w for w_0 .

Continuing Lord Rayleigh's method of demonstration, if a flat circular diaphragm is clamped at its edge between a pair of flat circular rings, then, referring to (2), we have w vanishing at $r=a$, the clamping radius, and since there is to be no bending or slope of the diaphragm at the clamped boundary, we have also $(dw/dr)=0$ at $r=a$.

Entering (2) with $w=0$, we have:

$$\lambda = - \frac{J_0(ka)}{J_0(ika)} \quad \text{numeric.} \quad (3)$$

Also differentiating (2) with respect to r , for $r=a$, we obtain:

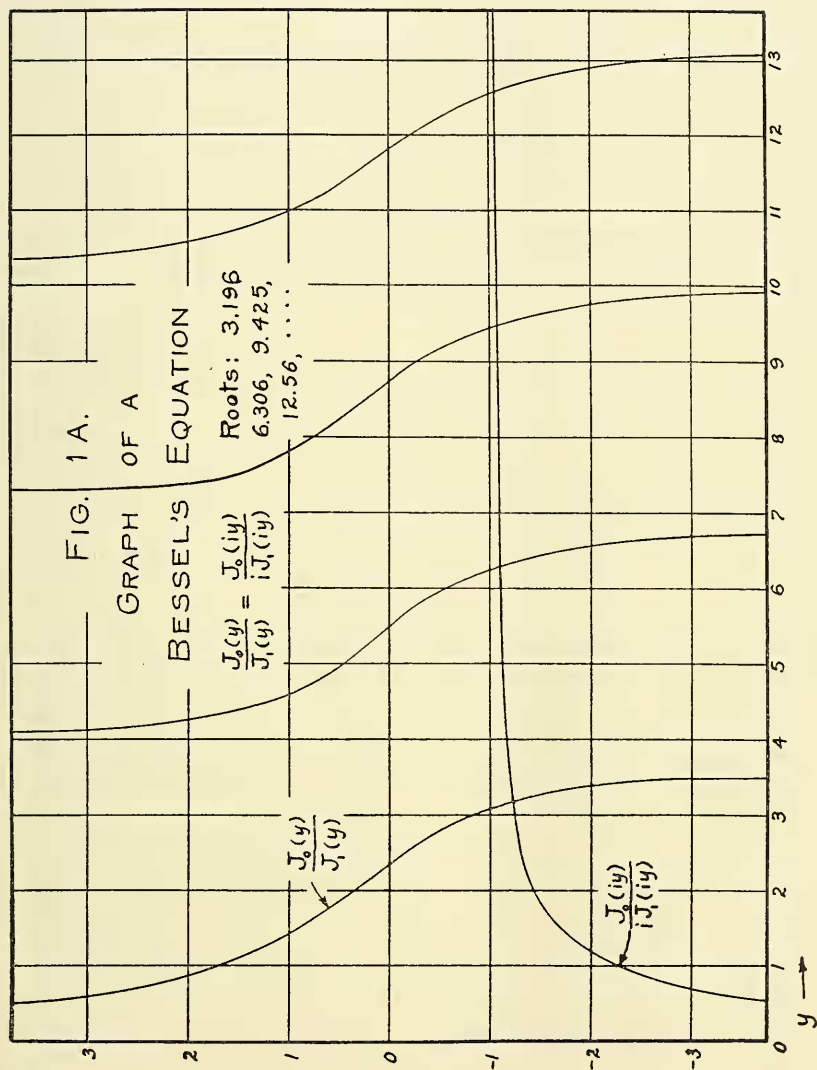
$$\frac{1}{k} \frac{dw}{dr} = J_0'(ka) + i\lambda J_0'(ika) = 0 \quad \text{numeric,} \quad (4)$$

whence

$$\lambda = - \frac{J_0'(ka)}{iJ_0'(ika)} \quad \text{numeric.} \quad (5)$$

Combining (3) and (5) we obtain:

$$\frac{J_0(ka)}{J_0(ika)} = \frac{J_0'(ka)}{iJ_0'(ika)} = \frac{J_1(ka)}{iJ_1(ika)} \quad \text{numeric.} \quad (6)$$



This is a transcendental equation involving Bessel's Functions of the zeroth and first orders. It is capable of being satisfied, by trial, with an indefinitely great number of roots, each corresponding to a possible mode of vibration with nodal circles. Fig. 1*A* indicates graphically the method of determining the successive roots of (6). The points of intersection of the lower curve with the successive descending branches, indicate the values of $y = kr$ which satisfy (6). In order to have the fundamental mode of vibration, there must be no nodal circles, which means that the first and lowest root for ka must be taken in (6). This root is at $ka = 3.196$ Placing this value for ka in (3) we have:

$$\lambda = \frac{-J_0(3.196)}{J_0(i3.196)} = -\frac{-0.3197}{5.730} = +0.05571 \quad \text{numeric. (7)}$$

Re-entering (2) with this value of λ , we have for the fundamental mode of vibration of the circular diaphragm:

$$w_{\max} = P\{J_0(kr) + 0.05571J_0(ikr)\} \quad \text{cm. (8)}$$

In Fig. 1*B*, the abscissas correspond both to kr , where $k = 1.21 \text{ cm.}^{-1}$, and to r in cm., the relation being as already pointed out that at the boundary $r = a = 2.62 \text{ cm.}$ and $kr = 3.196$. The ordinates are the numerical values of Bessel's functions as taken from Tables. They also represent vibratory amplitudes of the diaphragm, taking the maximum amplitude at the center ($r = 0$) in microns, corresponding to the heavy curve. The upper faint curve shows the graph of the first Bessel function $J_0(kr)$; while the lower faint curve shows the corresponding graph of λ times the second Bessel function, or $0.05571J_0(ikr)$. Adding these two graphs, as called for by (2), we obtain the heavy curve, which represents the theoretical amplitude of vibration along any radius of this particular diaphragm, assuming such a scale that 1.056 corresponds to the maximum or central amplitude. The small circles near this curve show the amplitudes observed with the aid of the vibration explorer.

APPENDIX II.

Elementary Theory of the Steady Vibration Amplitude of a Diaphragm Vibrating in its Fundamental Mode, as a Function of the Impressed Frequency.

Let w = the vibration amplitude at the center of the diaphragm⁸
(cm. $\angle$),

w_r = the vibration amplitude at the radius r (cm. $\angle$),

$\dot{w}$ = the vibration velocity at the center of the diaphragm
(cm./sec. $\angle$),

$\ddot{w}$ = the vibration acceleration at the center of the diaphragm
(cm./sec.² $\angle$),

r = frictional resistance to motion of the diaphragm, referred to the equivalent mass, see below (dynes/cm. per sec. $\angle$),

t = elapsed time from a given epoch (seconds),

s = elastic force of the diaphragm per cm. of displacement, referred to the equivalent mass (dynes per cm. $\angle$),

$f = Fe^{i\omega t}$ = impressed simple harmonic moving force on the diaphragm tending to produce displacement w , and measured in the direction of w , referred to the equivalent mass (dynes $\angle$),

$i = \sqrt{-1}$,

$\omega = 2\pi n$ = the angular velocity of a simple harmonic motion of frequency n (radians/sec.),

m = equivalent mass of the diaphragm, defined by the condition that the energy of motion of this mass with the velocity $\dot{w}$ at the center, is equal to the actual energy of the diaphragm with its distributed mass and velocities, according to the equation:

$$\frac{m}{2} (\dot{w})^2 = \frac{2\pi\rho'}{2} \int_0^a r (\dot{w}_r)^2 dr \quad \text{ergs, (1)}$$

where ρ' = superficial density of the diaphragm (gm./cm.²),

$$m = \frac{2\pi\rho'}{w_{\max}^2} \int_0^a (w_r)^2 r dr \quad \text{gm., (2)}$$

⁸ The sign $\angle$ after a unit indicates a "complex quantity."

since the velocities $\dot{w}$ and $\dot{w}_r$ being assumed simply harmonic, are respectively proportional to their maximum displacements $w_{\max}$ and w_r .

Then on the assumptions that the diaphragm vibrates like its equivalent mass collected at the center, with its observed central velocity, with an elastic opposing force $s\dot{w}$ on this mass, proportional to the displacement, and with a resisting force $r\dot{w}$ on this mass proportional to the velocity, then the equation of motion of the diaphragm in terms of equivalent mass will be⁹

$$s\dot{w} + r\dot{w} + m\ddot{w} = f = F\epsilon^{i\omega t} \quad \text{dynes } \angle. \quad (3)$$

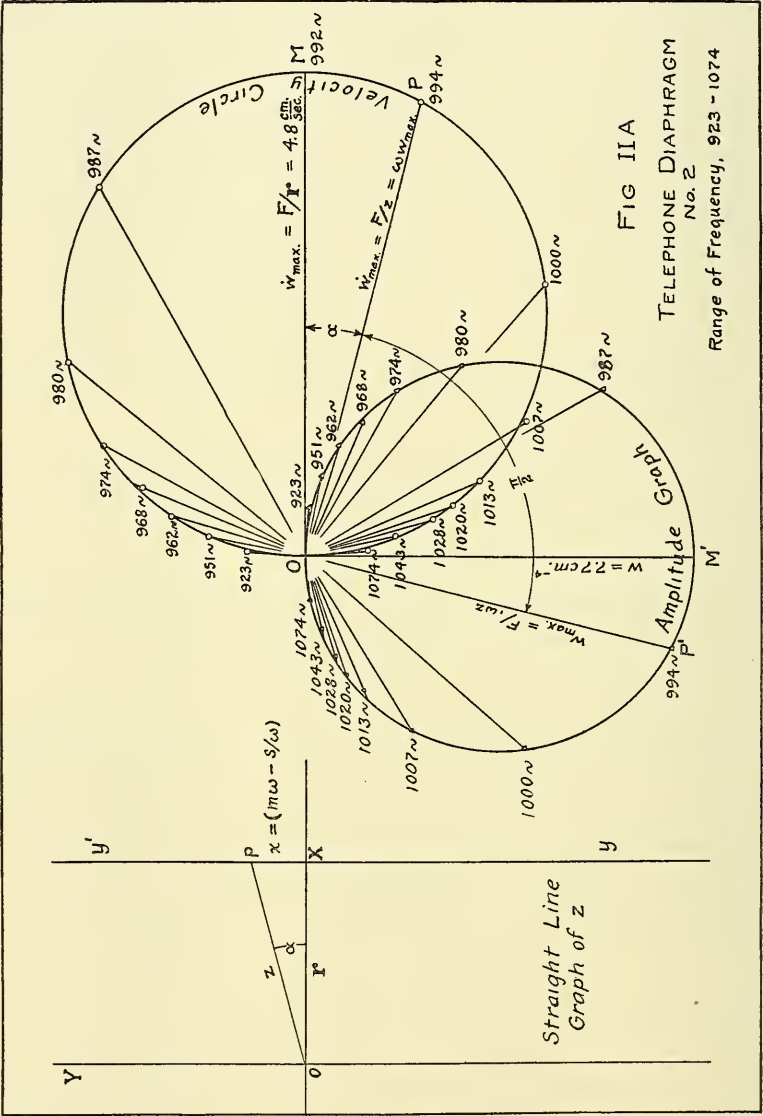
The solution of this equation, in terms of velocity $\dot{w}$, and the steady state, is known to be

$$\dot{w} = \frac{f}{r + i\left(m\omega - \frac{s}{\omega}\right)} = \frac{f}{r + ix} = \frac{f}{z} = \frac{F}{z}\epsilon^{i\omega t} \quad \frac{\text{cm.}}{\text{sec.}} \angle, \quad (4)$$

where x is the "mechanical reactance," and z is the complex "mechanical impedance," by analogy to alternating electric current theory. Both x and z have the same dimensions as r .

The mechanical impedance relations are indicated in Fig. II*A* at the left-hand side. OX and OY being rectangular coördinates, the "mechanical resistance" r in dynes per unit velocity, is measured along OX , and is assumed to remain constant at all frequencies. As the frequency n is increased (and with it the vibratory angular velocity ω) from zero to infinity, the reactance $x = (m\omega - s/\omega)$ varies from $-\infty$ to $+\infty$ along the line yXy' . The mechanical impedance z which is the vector sum of r and ix , will be represented by a complex quantity, or plane vector Op , the extremity of which remains on the line yXy' . At the particular or resonant value of ω , for which $m\omega - s/\omega = 0$, the reactance vanishes, and the impedance z coincides with the resistance r . As shown in the figure, p lies above OX , corresponding to a value of ω somewhat greater than the critical or resonant value.

⁹ See Bibliography No. 8.



Equation (4) shows that the displacement velocity $\dot{w}$ is equal to the impressed vibro-motive force f , divided by the impedance z . The locus of this velocity, as ω varies from 0 to ∞ with constant F , becomes a circle OMP , the diameter OM of which is equal to F/r cm. per sec., while the angle α of the chord OP , measuring the velocity, is equal and of opposite sign to the angle α of the impedance z . In the case represented by Fig. II*A*, the telephone diaphragm No. 2 was actuated electromagnetically at constant alternating-current strength, under varying frequency. At the frequency $n=992 \sim$, the vibratory velocity $OM=4.8$ cm. sec., was a maximum, and was in phase with the impressed vibro-motive force F . At $n=994 \sim$, the mechanical impedance had increased to op at the angle $\alpha=14^\circ$, and the vibratory velocity had fallen from OM to OP or from 4.8 to 4.65 cm. per sec. lagging in phase behind the impressed vibro-motive force by 14° . The diagram shows that between the frequencies of 923 and 1,074 $\sim$, the vector displacement velocity $\dot{w}$ had moved over nearly the entire circumference of the velocity circle OMP , and from a phase nearly 90° ahead of the impressed vibro-motive force to nearly 90° behind it.

If we integrate (4) with respect to time, we obtain, for the steady state of motion,

$$w = \int \dot{w} dt = \int \frac{F\epsilon^{i\omega t}}{z} dt = \frac{F\epsilon^{i\omega t}}{i\omega z} = -\frac{iF\epsilon^{i\omega t}}{\omega z}, \quad \text{cm. } \angle. \quad (5)$$

This shows that the instantaneous displacement is ω times less than the corresponding instantaneous velocity, and is 90° behind it in phase. If we consider the maximum displacement, we have

$$w_{\max} = -\frac{iF}{\omega z} \quad \text{cm.} \quad (6)$$

The locus of $w_{\max}$ is therefore a closed curve distorted from a circle by the effect of varying ω in the denominator. Considering it as an approximate circle for this case, the diameter OM' corresponding to $n=992 \sim$ represents a displacement amplitude of 7.7μ , lagging approximately 90° behind the maximum velocity OM . At the frequency $994 \sim$, the displacement would be $OP'=7.48 \mu$,

lagging 90° behind OP . As the frequency varies between 923 and 1,074 $\sim$, the displacement amplitude almost covers the entire graph of the approximate circle OMP' , commencing at about 1μ , nearly in phase with the vibro-motive force, and ending at about 1μ in nearly opposite phase. These amplitudes correspond to the ordinates of the resonance curve in Fig. 9.

It follows from (4) that if the vibro-motive force f is kept constant, and the angular velocity adjusted until the central vibration velocity is a maximum, this will occur when the mechanical reactance is zero, or when

$$m\omega_0 - \frac{s}{\omega_0} = 0 \quad \frac{\text{dynes}}{\text{cm./sec.}}, \quad (7)$$

that is

$$\omega_0 = \sqrt{\frac{s}{m}} \quad \frac{\text{radians}}{\text{sec.}} \quad (8)$$

So that

$$s = m\omega_0^2 \quad \frac{\text{dynes}}{\text{cm.}} \quad (9)$$

When the vibro-motive force f is made to vanish in (3) with the diaphragm in motion, the solution of the equation is

$$w = W\epsilon^{-\frac{rt}{2m}} \sin(\omega t + e) \quad \text{cm.}, \quad (10)$$

where W is the initial displacement (cm), and e a suitable phase (radians). If we obtain two successive values of w , (w_1 and w_2), corresponding to two successive elongations in the same direction, we have

$$\frac{w_1}{w_2} = \epsilon^{r/2mn} = \epsilon^{\Delta/n} \quad \text{numeric}, \quad (11)$$

whence

$$r = 2mn \log_e (w_1/w_2), \quad \text{dynes}/(\text{cm./sec.}), \quad (12)$$

where Δ is the damping constant ($1/\text{sec.}$).

The quantity $\log_e (w_1/w_2)$ is well known as the logarithmic decrement of the decay curve.

APPENDIX III.

Elementary Theory of Equivalent Mass.

In (2) of Appendix II., the expression for equivalent mass m is

$$m = \frac{2\pi\rho'}{v_{\max}^2} \int_0^a v_r^2 \cdot r \, dr \quad \text{gm.} \quad (1)$$

or m is the mass which, vibrating at the center of the diaphragm with the observed maximum amplitude $v_{\max}$, would have the same kinetic energy as the total distributed kinetic energy of the diaphragm.

In order, therefore, to determine the equivalent mass of a diaphragm, it is necessary to integrate r times the square of the amplitude over its surface. Assuming that the vibration follows Rayleigh's Bessel-function theory as outlined in Appendix I., it should be sufficient to integrate $v_r^2 \cdot r$ over the surface, mathematically. We are indebted to Dr. Geo. A. Campbell for an indication of the solution of this integral.¹⁰

In (1)

$$v_{\max} = P[J_0(0) + \lambda J_0(i0)] = P(1 + \lambda) \quad \text{cm.} \quad (2)$$

by reference to (8) Appendix I., putting $r=0$.

Also

$$v_r = P[J_0(kr) + \lambda J_0(ikr)] \quad \text{cm.} \quad (3)$$

$$\therefore m = \frac{2\pi\rho'}{P^2(1 + \lambda)^2} \int_0^a P^2 \{ J_0^2(kr) + \lambda^2 J_0^2(ikr) + 2\lambda J_0(kr)J_0(ikr) \} r dr \quad (4)$$

$$\begin{aligned} &= \frac{2\pi\rho'}{(1 + \lambda)^2} \left[\int_0^a J_0^2(kr) r \cdot dr + \int_0^a \lambda^2 J_0^2(ikr) r \cdot dr \right. \\ &\quad \left. + \int_0^a 2\lambda J_0(kr)J_0(ikr) r \cdot dr \right] \\ &= \frac{2\pi\rho'}{(1 + \lambda)^2} \left[\frac{a^2}{2} \{ J_0^2(ka) + J_1^2(ka) \} \right. \\ &\quad \left. + \frac{\lambda^2 a^2}{2} \{ J_0^2(ika) + J_1^2(ika) \} \right. \\ &\quad \left. + \frac{2\lambda a}{k^2 - i^2 k^2} \{ kJ_0(ika)J_1(ka) - ikJ_0(ka)J_1(ika) \} \right], \quad (5) \end{aligned}$$

¹⁰ Bibliography (11), (12), (13).

where

$$J_0^2(kr) \text{ stands for } \{J_0(kr)\}^2.$$

But $M = \pi \rho' a^2$ is the total mass of the vibrating diaphragm area.

$$\therefore \frac{m}{M} = \frac{1}{(1 + \lambda)^2} \left[\{J_0^2(ka) + J_1^2(ka)\} + \lambda^2 \{J_0^2(ika) + J_1^2(ika)\} + \frac{2\lambda}{ak} \{J_0(ika)J_1(ka) - iJ_0(ka)J_1(ika)\} \right]. \quad (6)$$

Applying the ratios of (6) Appendix I., this reduces to:

$$\begin{aligned} \frac{m}{M} &= \frac{1}{(1 + \lambda)^2} \cdot 2J_0^2(ka) \\ &= \frac{1}{(1.05571)^2} \cdot 2J_0^2(3.196) \\ &= \frac{0.20378}{1.1145} \\ &= 0.18285 \end{aligned}$$

or, to three significant digits, 0.183.

The "equivalent mass coefficient," 0.183, for this diaphragm, had also been obtained by quadrature methods applied to the heavy curve in Fig. 1B, before the integration was performed as above.

In the case of steel telephone diaphragms excited by bipolar electromagnets, the curves of w_r , r are likely to depart from simple Bessel-function curves, see Fig. 14. In such cases, the coefficient of equivalent mass must be deduced from the exploration curve. In cases examined, this coefficient varied between 0.2 and 0.5.

A quadrature method employed to find the equivalent mass coefficient from curves of any shape is as follows:

Draw the w_r curve as in Fig. 1B. Divide the line of abscissas into an integral number n of annular rings of equal area; so that each ring will have a mass of M/n , where M is the total mass of the circular vibrating area of the diaphragm, in grams. We then multiply this annular mass into the square of the observed amplitudes at the middle points of the successive annuli. The sum of these terms will be equal to the product of the equivalent mass m ,

TABLE IV.

	kr	w	w (ave.)	w^2 (ave.)
	.0000	1.0557		(1.1145)
1	.4511	1.008	1.032	1.0650
2	.6380	.962	.985	.9702
3	.7814	.918	.940	.8836
4	.9023	.875	.897	.8046
5	1.009	.833	.854	.7293
6	1.105	.792	.812	.6593
7	1.194	.752	.772	.5960
8	1.276	.713	.732	.5358
9	1.353	.677	.695	.4830
10	1.427	.641	.659	.4343
11	1.496	.606	.624	.3894
12	1.563	.572	.589	.3469
13	1.627	.539	.555	.3080
14	1.688	.507	.523	.2735
15	1.747	.477	.492	.2421
16	1.805	.449	.463	.2144
17	1.860	.421	.435	.1892
18	1.914	.394	.408	.1665
19	1.966	.367	.380	.1444
20	2.018	.341	.354	.1253
21	2.067	.318	.330	.1089
22	2.116	.295	.307	.0942
23	2.163	.273	.284	.0807
24	2.210	.251	.262	.0686
25	2.256	.232	.242	.0586
26	2.300	.213	.223	.0497
27	2.344	.195	.204	.0416
28	2.387	.178	.186	.0346
29	2.429	.162	.170	.0289
30	2.471	.146	.154	.0237
31	2.512	.131	.139	.0193
32	2.552	.117	.124	.0154
33	2.592	.104	.110	.0121
34	2.631	.092	.098	.0096
35	2.669	.080	.086	.0074
36	2.707	.070	.075	.0056
37	2.744	.060	.065	.0042
38	2.781	.050	.055	.0030
39	2.817	.041	.045	.0020
40	2.853	.033	.037	.0014
41	2.889	.026	.030	.0009
42	2.924	.021	.023	.0005
43	2.958	.016	.019	.0004
44	2.992	.012	.014	.0002
45	3.026	.009	.011	.0001
46	3.060	.006	.008	.00006
47	3.093	.004	.005	.00002
48	3.126	.002	.003	.000009
49	3.158	.001	.001	.000001
50	3.196	.000	.000	.000000
				10.2325

$$m = (M/50)(10.2325/1.1145) = .183 M.$$

and the square of the maximum observed amplitude at the center, or

$$m = \frac{M \sum w_r^2}{n w_{\max}^2} \quad \text{gm.} \quad (7)$$

The preceding table sets forth this process for the curve of Fig. 1B, drawn theoretically, and checked observationally, with $n = 50$, or the diaphragm divided into 50 annuli of equal mass. The result is that the equivalent mass is 18.3 per cent. of the actual mass of the vibrating area. This result checks that obtained from the mathematical integration of the Bessel curve.

Although 50 annuli of equal area and mass were taken in the case above worked out, so as to attain a fairly high degree of precision in the evaluated equivalent-mass coefficient; yet, for many purposes, a sufficient degree of precision might be attained by taking only 10 such equal annular areas.

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TABLE OF SYMBOLS.

- a = Radius of the diaphragm clamping-circle (cm.),
 α_n = A phase angle measured around the diaphragm (radians),
 b = Thickness of the diaphragm (cm.),
 c = A constant of the material of the diaphragm (cm./second $\frac{1}{2}$),
 d = Sign of differentiation
 Δ = Damping constant = $n \log_e (w_1/w_2) = r/2m$ (second⁻¹),
 e = Time-phase (radians),
 ϵ = Napierian logarithmic base (numeric),
 $f = F\epsilon^{i\omega t}$ Impressed simple harmonic moving force on the diaphragm (dynes) $\angle$
 f_s = Statical tension (dynes),
 F = Maximum value of a vibratory force (dynes),
 $i = \sqrt{-1}$ (numeric),
 J_n = A Bessel's Function of the n th order (numeric),
 J' = The first derivative of J with respect to r (numeric),
 k = A constant of the material of the diaphragm, defined by $k = (\sqrt{\omega})/c$ (cm.⁻¹),
 L = Distance from mirror to scale of explorer (cm.),
 l = Radius arm of small mirror in explorer (cm.),
 λ = A constant satisfying boundary conditions (numeric),
 M = Total mass of diaphragm (in Appendix III) (gm.),
 M = Magnification factor of explorer (numeric),
 m = Equivalent mass of the diaphragm (gm.),
 μ = Micron, 10^{-4} cm. (cm.⁻⁴),
 n = Frequency of diaphragm vibration (cycles/second),
 n_0 = Resonant frequency of diaphragm vibration (cycles/sec),
 n = Number of annular rings in equivalent mass theory of App. III (numeric),
 n (Subscript) = Number of nodal diameters (order of Bessel's Function) (numeric),
 P = Constant of amplitude-magnitude (cm.),
 $\pi = 3.1416$ (numeric),
 ϕ = Angle in the explorer between the plane of mirror and plane of diaphragm (deg.),
 q = Young's modulus for diaphragm material (dynes/cm.²),
 r = Frictional resistance to motion of diaphragm $\frac{\text{dynes}}{\text{cm./sec.}}$,

- r = Distance along a radius (cm.),
 ρ = Density of diaphragm material (gm./cm.³),
 ρ' = Superficial density of diaphragm (gm./cm.²),
 s = Elastic force of diaphragm per centimeter of displacement,
 referred to equivalent mass (dynes/cm.),
 σ = Poisson's ratio for material of diaphragm (numeric),
 Σ = Sign of summation,
 t = Time elapsed from a given epoch (seconds),
 θ = Azimuth angle measured on surface of diaphragm (radians)
 vmf. = Vibro-motive force (dynes) $\angle$,
 W = Initial displacement in a vibratory motion (cm.),
 w and w_0 = Amplitude of a point on surface of diaphragm for fun-
 damental mode of vibration (cm.) $\angle$,
 w_r = Amplitude of vibration of a point at radius r from center of
 diaphragm (cm.) $\angle$,
 w_n = Instantaneous amplitude of vibration (cm.),
 $w_{\max}$ = Maximum cyclic amplitude at center (cm.),
 $\dot{w}$ = Vibratory velocity at center of diaphragm (cm./sec.) $\angle$,
 $\ddot{w}$ = Vibratory acceleration at center of diaphragm (cm./sec.²) $\angle$,
 w_s = Statical displacement of center of diaphragm (cm.),
 $ix = i(m\omega - s/\omega)$ "Mechanical reactance" of vibrating diaphragm
 (by analogy to alternating-current theory) {dynes/(cm./
 sec.)} $\angle$,
 $z = (r + ix)$ "Mechanical impedance" of vibrating diaphragm
 (by analogy to alternating-current theory) {dynes/(cm./
 sec.)} $\angle$,
 $\omega = 2\pi n$ = Angular velocity of vibratory motion (radians/sec.),
 $\omega_0 = 2\pi n_0$ = Angular velocity at resonance (radians/sec.),
 ∞ = Infinity,
 $\angle$ = This sign after a unit indicates a "complex quantity,"
 $\sim$ = Cycles or vibrations per second (cycles/sec.).